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A note on T_0 Domination

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Abstract- A set $D \subseteq V$ of a graph $G(V,E)$ is called a dominating set if every vertex in G is either in D or is adjacent to an element of D. A simple graph G is said to be T_0, if for any two distinct vertices u and v of G, either one of u and v is isolated or there exists an edge e such that either e is incident with u but not with v or e is incident with v but not with u. If $\langle D \rangle$ of a dominating set D of the graph G is a T_0 graph, then it is called a T_0 dominating set and if $\langle D \rangle$ is both connected and T_0, then it is called a connected T_0 dominating set. The minimum cardinality of all T_0 dominating sets and connected T_0 dominating sets are respectively called T_0 domination number and connected T_0 domination number and are denoted respectively by $\gamma_{T_0}(G)$ and $\gamma_{cT_0}(G)$. In this paper T_0 domination number and connected T_0 domination number are introduced and some results on these new parameters are established.

Keywords- Domination number, T_0 domination number, connected T_0 domination number

1 INTRODUCTION

Graphs $G = (V(G), E(G))$ discussed in this paper are finite, simple and undirected. Any undefined term in this paper may be found in [1,4]. The degree [1] of a vertex v in graph G is denoted by $d_G(v)$ (or $d(v)$ if no specification of the graph G is needed), which is the number of edges incident with v. The maximum degree of G is denoted by $\Delta(G)$. The complement $\overline{G}$ of graph G [5] has $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if uv is not in $E(G)$. For a graph G, the number of vertices is called the order [5] of G and is denoted by $o(G)$. An empty graph [1] is a graph with no edges. An isolated vertex [4] is one whose degree is zero. A vertex in a graph is called a pendant vertex [6] if its degree is one. Any vertex adjacent to a pendant vertex is called a support vertex. A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph [1]. A complete graph on n vertices is denoted by K_n. A bipartite graph G is one whose vertex set can be partitioned into two subsets X and Y so that each edge has its ends in X and Y respectively. Such a partition (X,Y) is called a bipartition of G. A complete bipartite graph [1] is a simple bipartite graph with bipartition (X,Y) in which every vertex of X is joined to every vertex of Y.

The complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. The graph H is said to be an induced sub graph [2] of the graph G if $V(H) \subseteq V(G)$ and two vertices in H are adjacent if and only if they are adjacent in G. A tree [1] is a connected acyclic graph. A cut edge [1] of a graph G is an edge such that whose removal makes the graph disconnected. The open neighborhood [5] of v in $V(G)$ consists of those vertices adjacent to v in G and it is denoted by $N(v)$. The closed neighborhood

[5] of v is $N[v]=N(v)\bigcup\{v\}$. A matching [1] in a graph is a set of pair wise nonadjacent edges. Let $G=(V,E)$ be a graph. A set $D\subseteq V$ is called a dominating set [5] if every vertex in G is either in D or is adjacent to an element of D. The minimum cardinality of all dominating sets in G is called the domination number and is denoted by $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A detailed survey can be found in [5]. A dominating set D is called an independent dominating set [3] if $\langle D\rangle$ is the empty graph. A dominating set D is called a connected dominating set [7] if $\langle D\rangle$ is connected. The corresponding minimum cardinality of independent dominating set and connected dominating set are respectively called independent domination number and connected domination number and are denoted respectively by $i(G)$ and $\gamma_c(G)$.

In [8], V Seena and Raji Pilakkat defined the T_0 Graph as follows. A simple graph G is said to be T_0, if for any two distinct vertices u and v of G, one of the following hold

1. At least one of u and v is isolated.
2. There exists an edge e such that either e is incident with u but not with v or e is incident with v but not with u.

In this paper, a new domination parameter, T_0 domination number is introduced and some of its properties are studied. A T_0 dominating set is a dominating set $D\subseteq V$ such that $\langle D\rangle$ is T_0. Also it is proved that every independent dominating set in a graph is T_0 dominating. So that every graph has a T_0 dominating set. Hence the property of T_0 domination is applicable to all simple graphs.

2 T_0 Domination

T_0 domination is defined as follows.

Definition 2.1. Let G be any finite undirected simple graph with vertex set V. A dominating set $S\subseteq V$ is said to be T_0 dominating if $\langle D\rangle$ is a T_0 graph. The minimum cardinality of all such T_0 dominating sets is called T_0 domination number and is denoted by $\gamma_{T_0}(G)$. Such a T_0 dominating set with cardinality $\gamma_{T_0}(G)$ is called a γ_{T_0} -set.

Seena V and Raji P [8] proved that a graph G is T_0 if and only if K_2 is not a component of G. A characterization property of a T_0 dominating set follows directly from this result.

Theorem 2.1. Let $G=(V,E)$ be any graph. A dominating set $S\subseteq V$ is a T_0 dominating set if and only if no component of $\langle S\rangle$ is K_2.

Theorem 2.2. For any graph G, every independent dominating set is T_0 dominating.

Proof. Let $I\subseteq V$ be an independent dominating set of a graph $G=(V,E)$. Since K_2 is not a component of $\langle I\rangle$, $\langle I\rangle$ is a T_0 graph. $\square$

Corollary 2.3. For any graph G, $\gamma(G)\le\gamma_{T_0}(G)$

Remark 2.4. The converse of Theorem 2.2 need not be true. For example the set of all darkened vertices shown in figure 1 is T_0 dominating but not independent. Here $\gamma_{T_0}(G)=3$ and $i(G)=5$.

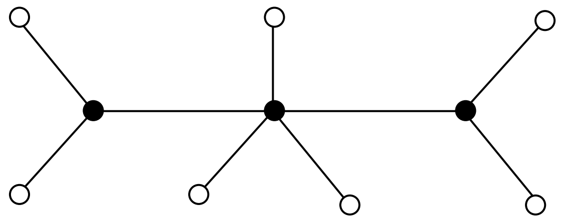

Figure 1- G

Theorem 2.5. For any positive integer k, there exist a graphs G such that $i(G)-\gamma_{T_0}(G)=k$

Proof. Consider the path P_3. Let G be the graph obtained from P_3 by attaching exactly j pendant edges to each vertex of P_3,where $j\ge 2$. Then $\gamma_{T_0}(G)=3$ and $i(G)=3+(j-1)$ when $j\ge 2$, Therefore $i(G)-\gamma_{T_0}(G)=j-1$. Since $j\ge 2$, $i(G)-\gamma_{T_0}(G)=k,\quad k=1,2,3...$ $\square$

Theorem 2.6 characterizes graphs $\gamma_{T_0}(G)=1$ $\gamma_{T_0}(G)=2$, $\gamma_{T_0}(G)=n-1$ and $\gamma_{T_0}(G)=n$.

Theorem 2.6 Let G be any graph on n vertices. Then

1. $\gamma_{T_0}(G)=1$ if and only if $\Delta(G)=n-1$
2. $\gamma_{T_0}(G)=2$ if and only if $i(G)=2$
3. $\gamma_{T_0}(G)=n$ if and only if $G=\overline{K}_n$
4. $\gamma_{T_0}(G)=n-1$ if and only if $G\cong K_2$ or $G\cong K_2\cup\overline{K}_{n-2}$

Proof. (1) is obvious.
(2) Suppose that $\gamma_{T_0}(G)=2$. Let $D\subseteq V(G)$ be a γ_{T_0} -set. Then $|D|=2$ and $\langle D\rangle$ is empty. Hence D is independent dominating. Also since $\gamma_{T_0}(G)\le i(G)$, it follows that $i(G)=2$. Conversely, let $i(G)=2$. If $\gamma_{T_0}(G)\ne 2$, then by Corollary 2.3, $\gamma_{T_0}(G)=1$. Then the γ_{T_0} -set is also independent dominating, contradicting i(G) = 2. Hence $\gamma_{T_0}(G)=2$.
(3) Let $\gamma_{T_0}(G)=n$. Then every γ_{T_0} -set D contains every vertices of G and hence $\langle D\rangle=G$. Also since $\gamma_{T_0}(G)\le i(G)$ and $o(G)=n$, $i(G)$ must be n. Hence $G=\overline{K_n}$. The converse is obvious.

(4) If $G=K_2$ or $K_2\cup\overline{K}_{n-2}$, then $\gamma_{T_0}(G)=n-1$.
Conversely suppose that $\gamma_{T_0}(G)=n-1$.
Case (i) G is connected. If $\Delta(G)=0$, then since G is connected, $G\cong K_1$ and therefore $\gamma_{T_0}(G)=1=n$. If $\Delta(G)=1$, then since G is connected, G has exactly two vertices and $G\cong K_2$.

Therefore $\gamma_{T_0}(G)=n-1$. If $\Delta(G)\ge 2$, then $i(G)\le n-\Delta(G)$ and hence we have the following
$\gamma_{T_0}(G)\le n-\Delta(G)\le n-1$, by Corollary 2.3.
Therefore K_2 is the only connected graph with $\gamma_{T_0}(G)=n-1$.

Case (ii) G is disconnected. If $\Delta(G)=0$ or if $\Delta(G)\ge 2$, then $\gamma_{T_0}(G)=n$ or less than or equal to $n-\Delta(G)$ respectively. Therefore, $\gamma_{T_0}(G)=n-1$ if and only if $\Delta(G)=1$. In this case, the only nontrivial connected component of G are K_2. Suppose that r components of G are K_2. Then we have, $1\le r\le\left\lceil\frac{n}{2}\right\rceil$ and $\gamma_{T_0}(G)=n-r$. Thus $\gamma_{T_0}(G)=n-1$ if and only if $n-r=n-1$. ie., if and only if $r=1$. $\square$

Corollary 2.7. Let G be a graph of order n with $\Delta(G)>0$. If G is distinct from any of the graphs $K_2\cup nK_1$ where $n=1,2,3\ldots$ then $\gamma_{T_0}(G)\le n-2$. Further equality holds for $G=P_4$ and C_4

Theorem 2.8. Let G be any nontrivial connected graph of order n, then

$$\gamma_{T_0}(G)+\gamma_{T_0}(\bar{G})\le 2n-1 \qquad (1)$$

$$\gamma_{T_0}(G)\gamma_{T_0}(\bar{G})\le n(n-1) \qquad (2)$$

Further equality holds if and only if $G\cong K_2$.

Proof. If $G=K_1$ then $\gamma_{T_0}(G)=\gamma_{T_0}(\bar{G})=n$.
There are no nontrivial graphs for which $\gamma_{T_0}(G)=\gamma_{T_0}(\overline{G})=n$.
Therefore, $\gamma_{T_0}(G)+\gamma_{T_0}(\overline{G})\le 2n-1$
and $\gamma_{T_0}(G)\gamma_{T_0}(\overline{G})\le n(n-1)$.
Furthermore, equality holds in (1) and (2) if and only if either $\gamma_{T_0}(G)=n$ and $\gamma_{T_0}(\overline{G})=n-1$
or $\gamma_{T_0}(G)=n-1$ and $\gamma_{T_0}(\overline{G})=n$.
By Theorem 2.6, this is true if and only if $G\cong K_2$ or $\overline{G}\cong K_2$. $\square$

Theorem 2.9. For any graph G, if $i(G)=3$ then $\gamma_{T_0}(G)=3$

Proof. Let $i(G)=3$ and let $S\subseteq V(G)$ be a T_0 dominating set with $|S|<3$. Since a connected graph with two vertices is not T_0, S is an independent dominating set, a contradiction. □

Remark 2.10. The converse of Theorem 2.9 need not be true. For example in figure.1, $\gamma_{T_0}(G)=3$ but $i(G)=5.$.

Theorem 2.11. For complete bipartite graphs $K_{m,n}$,

$$\gamma_{T_0}(K_{m,n})=\begin{cases}1 \text{ if } m=1 \text{ or } n=1\\ 2 \text{ if } m=2, n\geq 2 \text{ or } n=2, m\geq 2\\ 3 \text{ if } m>2, \quad n>2\end{cases}$$

Proof. Case (i) If either m or n is one, then $\Delta(G)=o(G)-1.$ Hence $\gamma_{T_0}(G)=1$ by Theorem 2.6 Case (ii) If one of m or n is exactly 2, then $i(G)=2.$ Hence by Theorem 2.6, $\gamma_{T_0}(G)=2$. Case (iii) Since $\gamma_{T_0}(K_{m,n})=2$, and since $\gamma(G)\leq\gamma_{T_0}(G)$ for any graph G, we have, $\gamma_{T_0}(K_{m,n})\geq 2$. Let U,V be the two partite set of $K_{m,n}$. If we take two vertices from the same partite set say U of $K_{m,n}$, then they will not dominate other vertices of U and if we take one vertex from U and other vertex from V then these two vertices dominate $K_{m,n}$ but the sub graph induced by these vertices is isomorphic to K_2, which is not a T_0 graph. Therefore, $\gamma_{T_0}(K_{m,n})\geq 3$. The choice of any two vertices from one partite set and a third vertex from the other one dominate $K_{m,n}$ and their span is P_3, which is T_0. Hence the theorem. □

Corollary 2.12. For $K_{m,n}$, $\gamma_{T_0}(K_{m,n})\leq 3$ for all values of m and n.

Remark 2.13. If $G=K_{m,n}$; $m\geq 4, n\geq 4$ then, $\gamma(G)<\gamma_{T_0}(G)<i(G)$.

Theorem 2.14. If G is a connected graph of order ≥ 2, which contain no K_3 as an induced subgraph, then $\gamma_{T_0}(\overline{G})=2$

Proof. Since G is a connected graph of order ≥ 2, it contains at least an edge say uv. If $o(G)=2$ then G is isomorphic to K_2 and $\overline{G}$ is isomorphic to $\overline{K}_2$ Therefore, $\gamma_{T_0}(\overline{G})=2$. If $o(G)>2$, then no vertex of G is adjacent to both u and v, because G is triangle free. Therefore every vertex in G which are adjacent to u are dominated by v in $\overline{G}$ and those vertices adjacent to v in G are dominated by u in $\overline{G}$ and all vertices which are non adjacent to both u and v are dominated by both u and v in G. So $\{u,v\}$ forms an independent dominating set of G. Therefore it is also a T_0 dominating set of $\overline{G}$. Hence $\gamma_{T_0}(\overline{G})\leq 2$. Now let if possible $\gamma_{T_0}(\overline{G})=1$, then G would have an isolated vertex, a contradiction. Which proves $\gamma_{T_0}(\overline{G})=2.$ □

Theorem 2.15. Let $G(V,E)$ be any graph. Then for any T_0 dominating set $D\subseteq V$ of G, $\langle D\rangle$ can never be a matching of G.

Proof. Suppose if possible, $D\subseteq V$ be a T_0-dominating set of G such that, $\langle D\rangle$ is a matching of G. Then $\langle D\rangle$ consists of disconnected edges. That is, $\langle D\rangle$ has K_2 as a component, a contradiction to D is a T_0-dominating set of G. □

3 Connected T_0 Domination.

Definition 3.1. Let $G=(V,E)$ be any graph. A dominating set $S\subseteq V$ is called a connected T_0 dominating set, if $\langle S\rangle$ is both connected and T_0. The minimum cardinality of all connected T_0 dominating sets is denoted by $\gamma_{cT_0}(G)$ and is called the connected T_0 domination number of G. Any connected T_0 dominating set with cardinality $\gamma_{cT_0}(G)$ is called a γ_{cT_0}-set of G.

Observation 3.1. For any connected graph G, $\gamma_c(G)\leq\gamma_{cT_0}(G)$. This inequality is sharp for P_4

Theorem 3.2. Let G be any connected graph with $\gamma_c(G)\neq 2$ then $\gamma_{cT_0}(G)=\gamma_c(G)$

Proof. Since $\gamma_c(G) \neq 2$, the graph induced by any γ_c-set is not K_2 and hence the γ_c-set is connected T_0 dominating. Also since $\gamma_c(G) \leq \gamma_{cT_0}(G)$, it follows that $\gamma_{cT_0}(G) = \gamma_c(G)$. □

Theorem 3.3. Let a and b be two positive integers with $a > 2$ and $b \geq 2a+2$. Then there is a graph G on b vertices with $\gamma(G) = \gamma_c(G) = \gamma_{cT_0}(G) = a$ and $i(G) \geq a+1$.

Proof. Consider the path $P = (u_1, u_2, ..., u_a)$ on a vertices. Let $b \geq 2a + r,\ r \geq 2.$ Let G be the graph obtained from P by attaching two or more pendant edges at u_1 and u_2 and one pendant edge at each $u_i, i \geq 3$. Let $v_i, i \geq 3$ be the pendant vertices attached to $u_i, i \geq 3$. Clearly $D = \{u_1, u_2, ..., u_a\}$ is a γ-set which is also a connected T_0 dominating set. Hence $\gamma(G) = \gamma_c(G) = \gamma_{cT_0}(G) = a$. Any independent dominating set of G of minimum cardinality will be one among the following. $\{u_i, v_3, v_4, ..., v_a\} \cup N(u_j)$ where u_i is the vertex of maximum degree among u_1 and u_2 and $N(u_j)$ is the open neighborhood of u_1 or u_2 with minimum cardinality such that $i \neq j$

or $\{u_1, u_3, v_4, u_5, v_6 ...\} \cup N(u_2)$

if $d(u_1) \geq d(u_2)$ and $|N(u_2)| \leq |N(u_1)|$

or $\{u_2, v_3, u_4, v_5, u_6 ...\} \cup N(u_1)$

if $d(u_2) \geq d(u_1)$ and $|N(u_1)| \leq |N(u_2)|$.

In all these cases the cardinality of the i-set is

(a − 1) + min $\{d(u_1), d(u_2)\}$,

where min $\{d(u_1), d(u_2)\}$ ≥ 2.

Therefore $i(G) \geq a+1$. □

Theorem 3.4. Let T be any tree of order $n, n \geq 4$. If T is not isomorphic to $K_{1,n-1}$ then $\gamma_{cT_0}(\overline{T}) = 3$.

Proof. Since T is not isomorphic to $K_{1,n-1}$ (a star graph), $\Delta(T) \leq n-2$. Consider the following cases.

Case (i) T is not a path.

Since T is not a path, it has at least three pendant vertices say v_1, v_2 *and* v_3. Therefore, $d_{\overline{T}}(v_i) = n-2, i = 1,2,3$. Also since T is not a star, the support vertex of at least one of the v_i will be different from support vertices of the other two. Therefore $\{v_1, v_2, v_3\}$ forms a dominating set of $\overline{T}$. In $\overline{T}$, $\langle\{v_1, v_2, v_3\}\rangle$, the graph induced by $\{v_1, v_2, v_3\}$ is either K_3 or P_3. Hence $\{v_1, v_2, v_3\}$ is a connected T_0 dominating set of $\overline{T}$. So that $\gamma_{cT_0}(\overline{T}) \leq 3$. Since T has no isolated vertices, $\gamma_{cT_0}(\overline{T})$ cannot be one.

Also since there are no connected T_0 dominating sets of cardinality two, $\gamma_{cT_0}(\overline{T}) \geq 3$. Hence it follows that $\gamma_{cT_0}(\overline{T}) = 3$.

Case (ii) T is a path P_n on n vertices, $n \geq 4$.

Let v_1 and v_2 be the pendant vertices and v_3 be any one of the support vertices.

Then $d_{\overline{T}}(v_i) = n-2, i = 1,2$

and $d_{\overline{T}}(v_3) = n-3$.

In $\overline{T}$, the subgraph induced by $\{v_1, v_2, v_3\}$ is P_3 and it forms a connected T_0 dominating set of $\overline{T}$. Hence by the same reasoning as in case (i), it follows that $\gamma_{cT_0}(\overline{T}) = 3$. □

Corollary 3.5. Let T be a tree of order > 1, then $\overline{T}$ has a connected T_0 dominating set if and only if T is not a star.

Proof. If T is a star on n > 1 vertices, then $\overline{T}$ is disconnected. Hence $\overline{T}$ cannot have a connected T_0 dominating set. Conversely, let T be a tree, which is not a star. Then $n \geq 4$. Therefore by Theorem 3.4, T has a connected T_0 dominating set. □

Proposition 3.6. Let G be a bi-star B(m, n) on p vertices, then $\gamma_{cT_0}(G) = p - \max\{m, n\}$

Theorem 3.7. [7] For any tree T of order p, the connected domination number of $T = p - e$, where e is the number of pendant vertices in T

Theorem 3.8. Let T be any tree. Then the γ_c -set and γ_{T_0} -set are the same if and only if T is not a bi-star

Proof. Let T be a tree on p vertices. Assume that the γ_c -set and γ_{T_0} -set are the same. Suppose if possible, T is a bi-star.
Then by Theorem 3.7, $\gamma_c(T) = p - e = 2$, where, e is the number of pendant vertices. So that the graph induced by the γ_c -set is K_2, which is not T_0.
By Proposition 3.6 and Theorem 3.7, if T is a bi-star, then $\gamma_c \ (T) \neq \gamma_{cT_0}(T)$. □

We conclude with a conjecture.

Conjecture 3.8. There are no simple graphs G for which $\gamma_{T_0}(G) > \gamma(G)$

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